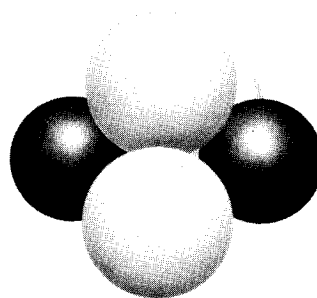


FIGURE 6-21

A schematic illustration of the NH_3 molecule. The light spheres represent the three H atoms arranged in a plane. The dark spheres represent two equivalent equilibrium positions of the single N atom.

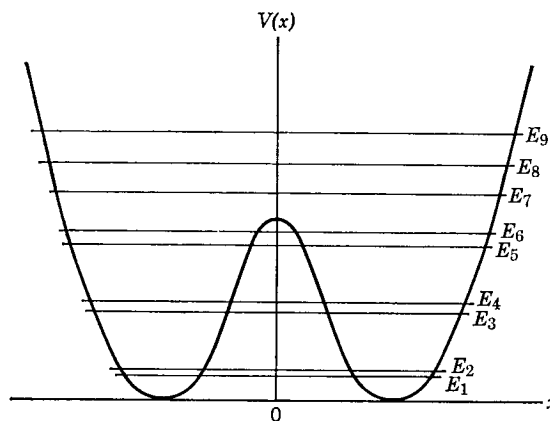


lower allowed energies of this binding potential are below the top of the barrier, as indicated in the figure. But penetration of the classically excluded region allows the N atom to tunnel through the barrier. If it is initially on one side, it will tunnel through and eventually appear on the other side. Then it will do it again in the opposite direction. The N atom actually oscillates slowly back and forth across the plane of the H atoms. The oscillation frequency is $\nu = 2.3786 \times 10^{10}$ Hz, when the molecule is in its ground state. This frequency is much lower than those found in molecular vibrations not involving barrier penetration, or in other atomic or molecular phenomena. Due to the resulting technical simplifications, the frequency was used as a standard in the first atomic clocks which measure time with maximum precision.

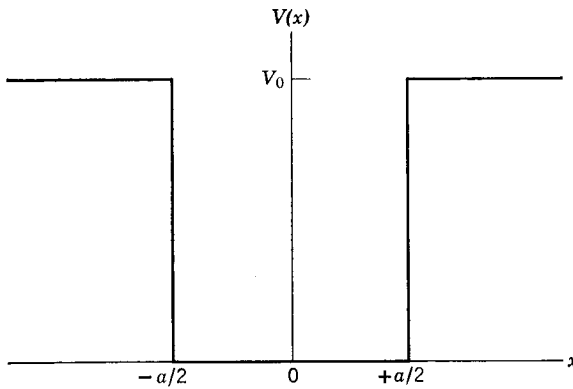
A recent, and very useful, example of barrier penetration of electrons is found in the *tunnel diode*. This is a semiconductor device, like a transistor, which is used in fast electronic circuits since its high frequency response is much better than that of any transistor. The operation of a tunnel diode will be explained in Chapter 13, in the context of a discussion of semiconductors. So here we shall say only that the device employs controllable barrier penetration to switch currents on or off so rapidly that it can be used to make an oscillator that can operate at frequencies above 10^{11} Hz.

6-7 The Square Well Potential

In the preceding sections we have treated the motion of particles in potentials which are not capable of binding them to limited regions of space. Although a number of

**FIGURE 6-22**

The potential energy of the N atom in the NH_3 molecule, as a function of its distance from the plane containing the three H atoms, which lies at $x = 0$. In its lower energy states, the total energy of the molecule lies below the top of the barrier separating the two minima, as indicated by the eigenvalues of the potential shown in the figure.

**FIGURE 6-23**

A square well potential.

interesting quantum phenomena showed up, energy quantization did not. Of course we know, from the qualitative discussion of the last chapter, that energy quantization can be expected only for potentials which are capable of binding a particle. In this section we shall discuss one of the simplest potentials having this property, the *square well potential*.

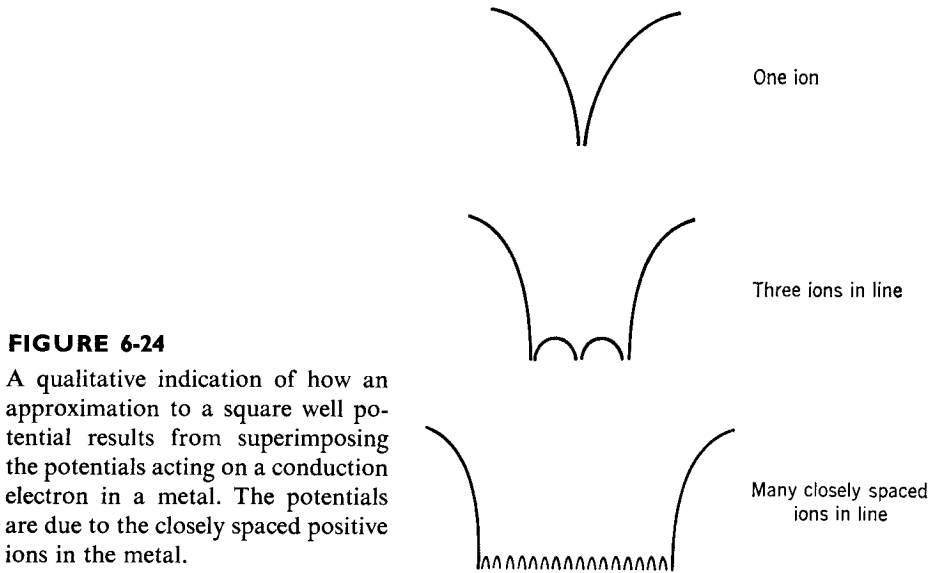
The potential can be written

$$V(x) = \begin{cases} V_0 & x < -a/2 \text{ or } x > +a/2 \\ 0 & -a/2 < x < +a/2 \end{cases} \quad (6-58)$$

The illustration in Figure 6-23 indicates the origin of its name. If the particle has total energy $E < V_0$, then in *classical mechanics* it can be only in the region $-a/2 < x < +a/2$ (within the well). The particle is bound to that region and bounces back and forth between the ends of the region with momentum of constant magnitude but alternating direction. Furthermore, *any* value $E \geq 0$ of the total energy is possible. But in *quantum mechanics* only *certain* discretely separated values of the total energy are possible.

The square well potential is often used in quantum mechanics to represent a situation in which a particle moves in a restricted region of space under the influence of forces which hold it in that region. Although this simplified potential loses some details of the motion, it retains the essential feature of binding the particle by forces of a certain strength to a region of a certain size. From the discussion in Example 6-2 it is apparent that it is a good approximation to represent the potential acting on a conduction electron in a block of metal by a square well. The depth of the square well is around 10 eV, and its width equals the width of the block. Figure 6-24 indicates, from a point of view different from that used in Example 6-2, how something like a square well can be obtained by superimposing the potentials produced by the closely spaced positive ions in the metal. In Example 6-3, we indicated that the motion of a neutron in a nucleus can be approximated by assuming that the particle is in a square well potential with a depth of about 50 MeV. The linear dimensions of the potential equal the nuclear diameter, which is about 10^{-14} m.

We begin our treatment by considering, qualitatively, the form of the eigenfunctions which are solutions to the time-independent Schroedinger equation for the square well potential of (6-58). As in the preceding sections, the problem decomposes itself into three regions: $x < -a/2$ (left of the well), $-a/2 < x < +a/2$ (within the well), and $x > +a/2$ (right of the well). The so-called *general solution* to the equation for the

**FIGURE 6-24**

A qualitative indication of how an approximation to a square well potential results from superimposing the potentials acting on a conduction electron in a metal. The potentials are due to the closely spaced positive ions in the metal.

region within the well is

$$\psi(x) = Ae^{ik_1x} + Be^{-ik_1x} \quad \text{where } k_1 = \frac{\sqrt{2mE}}{\hbar} \quad -a/2 < x < +a/2 \quad (6-59)$$

The first term describes waves traveling in the direction of increasing x , and the second term describes waves traveling in the direction of decreasing x . (This solution was derived in Section 6-2. If the student has not studied that section, he can easily show that it is a solution to the time-independent Schroedinger equation, for any values of the arbitrary constants A and B , by substituting it into (6-2).)

Now, the classical description of the particle bouncing back and forth within the well suggests that the eigenfunction in that region should correspond to an equal mixture of waves traveling in both directions. The two oppositely directed traveling waves of equal amplitude will combine to form a standing wave. We can obtain such behavior by setting the arbitrary constants equal to one another, so that $A = B$. This yields

$$\psi(x) = B(e^{ik_1x} + e^{-ik_1x})$$

which we write as

$$\psi(x) = B' \frac{e^{ik_1x} + e^{-ik_1x}}{2}$$

where B' is a new arbitrary constant defined by the relation $B' = 2B$. But this combination of complex exponentials gives us simply

$$\psi(x) = B' \cos k_1x \quad \text{where } k_1 = \frac{\sqrt{2mE}}{\hbar} \quad (6-60)$$

This eigenfunction describes a standing wave since inspection of the associated wave function $\Psi(x,t) = \psi(x)e^{-iEt/\hbar}$ shows that it has nodes in the fixed locations where $\cos k_1x = 0$.

We can also obtain a standing wave by setting $-A = B$. This gives

$$\psi(x) = A(e^{ik_1x} - e^{-ik_1x})$$

which we write as

$$\psi(x) = A' \frac{e^{ik_I x} - e^{-ik_I x}}{2i}$$

where A' is a new arbitrary constant defined by $A' = 2iA$. But this is just

$$\psi(x) = A' \sin k_I x \quad \text{where } k_I = \frac{\sqrt{2mE}}{\hbar} \quad (6-61)$$

Since both (6-60) and (6-61) specify solutions to the time-independent Schroedinger equation for the same value of E , and since that differential equation is linear in $\psi(x)$, their sum

$$\psi(x) = A' \sin k_I x + B' \cos k_I x \quad \text{where } k_I = \frac{\sqrt{2mE}}{\hbar} \quad -a/2 < x < +a/2 \quad (6-62)$$

is also a solution, as can be verified by direct substitution. In fact, this is a *general solution* to the differential equation for the region *within* the well because it contains two arbitrary constants—it is just as general as the solution of (6-59). Mathematically, the two are completely equivalent. However, (6-62) is more convenient to use in problems involving the motion of bound particles. Physically, (6-62) can be thought of as describing a situation in which a particle is moving in such a way that the magnitude of its momentum is known to be precisely $p = \hbar k_I = \sqrt{2mE}$, but the direction of the momentum could either be in the direction of increasing or decreasing x .

Now consider the solutions to the time-independent Schroedinger equation in the two regions *outside* the potential well: $x < -a/2$ and $x > +a/2$. In these regions the *general solutions* have the forms

$$\psi(x) = C e^{k_{II} x} + D e^{-k_{II} x} \quad \text{where } k_{II} = \frac{\sqrt{2m(V_0 - E)}}{\hbar} \quad x < -a/2 \quad (6-63)$$

and

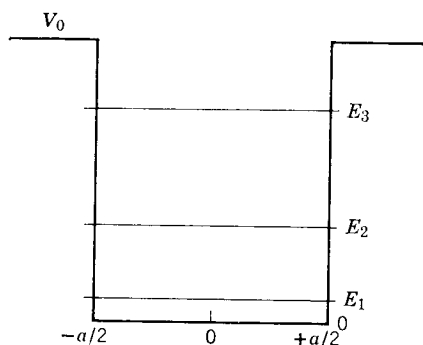
$$\psi(x) = F e^{k_{II} x} + G e^{-k_{II} x} \quad \text{where } k_{II} = \frac{\sqrt{2m(V_0 - E)}}{\hbar} \quad x > +a/2 \quad (6-64)$$

The two forms of $\psi(x)$ describe standing waves in the region outside the well, since in the associated wave function $\Psi(x, t) = \psi(x)e^{-iEt/\hbar}$ the x and t dependences occur as separate factors. These standing waves have no nodes, but they will be joined onto the standing waves inside the well which do have nodes. (The general solutions were derived in Section 6-3. Their validity, for any values of the arbitrary constants C , D , F , and G , can easily be verified by students who skipped that section by substitution in (6-13).)

Eigenfunctions valid for all x can be constructed by joining the forms assumed, in each of the three regions of x , by the general solutions to the time independent Schroedinger equation. These three forms involve six arbitrary constants: A' , B' , C , D , F , and G . Now since an acceptable eigenfunction must everywhere remain finite, we can immediately see that we must set $D = 0$ and $F = 0$. If this were not done the second exponential in (6-63) would make $\psi(x) \rightarrow \infty$ as $x \rightarrow -\infty$, and the first exponential in (6-64) would make $\psi(x) \rightarrow \infty$ as $x \rightarrow +\infty$. Four more equations involving the remaining arbitrary constants can be obtained by demanding that $\psi(x)$ and $d\psi(x)/dx$ be continuous at the two boundaries between the regions, $x = -a/2$ and $x = +a/2$, as is required for acceptable eigenfunctions. (They are already single valued.) But we cannot allow all four of the remaining arbitrary constants to be specified by these four equations. One of them must remain unspecified so that the

FIGURE 6-25

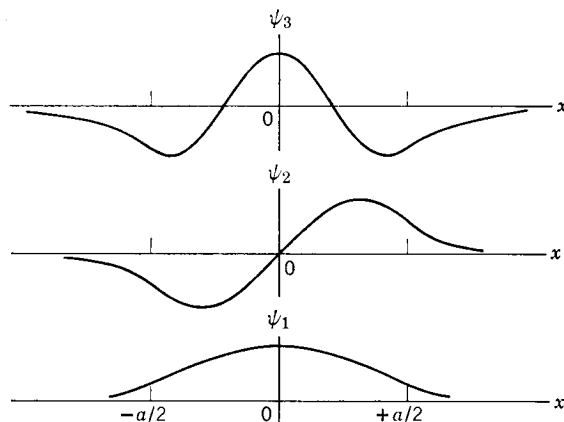
A square well potential and its three bound eigenvalues. Not shown is a continuum of eigenvalues of energy $E > V_0$.



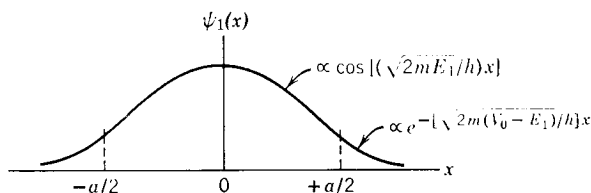
amplitude of the eigenfunction can be arbitrary. Arbitrary amplitude is required because the differential equation is linear in the eigenfunction $\psi(x)$. Thus there seems to be a discrepancy between the number of equations to be satisfied and the number of constants that can be adjusted, but it is resolved by treating the total energy E as an additional constant that can be adjusted, as needed. We shall find that this procedure works, but only for certain values of E . That is, there will emerge a certain set of possible values of the total energy E , and so the energy will be *quantized* to a set of eigenvalues. Only for these values of the total energy does the Schrodinger equation have acceptable solutions.

It is not difficult to carry through this procedure, as we shall see shortly in treating a special case. But the general case leads to a solution involving a complicated transcendental equation (an equation in which the unknown is contained in the argument of a function such as a sinusoid), which precludes expressing the solution mathematically in a concise way. Therefore, we relegate the details of the general solution to Appendix G, and here continue for a while with our qualitative discussion.

Figures 6-25 and 6-26 show, respectively, the eigenvalues and eigenfunctions for the three bound states of a particle in a particular square well potential. Not shown are a continuum of eigenvalues which extend from the top of the well on up, since any value of total energy E that is greater than the height of the potential walls V_0 is allowed. Also not shown are the continuum eigenfunctions. Focusing attention first on the eigenfunction that increases as the energy of the corresponding eigenvalue increases. As a consequence, the higher the energy of the eigenvalue the more numerous are the

**FIGURE 6-26**

The three bound eigenfunctions for the square well of Figure 6-25.

**FIGURE 6-27**

The first eigenfunction for a square well with walls of moderate height.

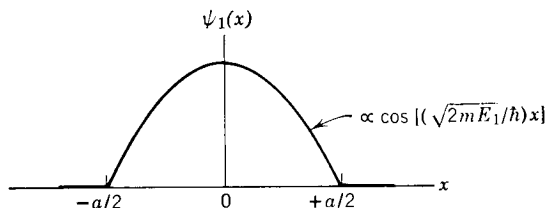
oscillations of the corresponding eigenfunction and the higher is its wave number. These results reflect the fact that the wave number k_I , in the solution of (6-62) for the region inside the well, is proportional to $E^{1/2}$. The square well potential depicted in the figure does not have a fourth *bound* eigenvalue because the associated value of k_I , and therefore of $E^{1/2}$, would be too large to satisfy the binding condition $E < V_0$.

Now consider the parts of the eigenfunctions that extend into the regions outside the well. In classical mechanics a particle could never be found in these regions since its kinetic energy is $p^2/2m = E - V(x)$, which is negative where $E < V(x)$. Note that the eigenfunctions go to zero in these *classically excluded regions* more rapidly the lower the energy of the corresponding eigenvalue. This agrees with the fact that the exponential parameter k_{II} , in the solutions of (6-63) and (6-64) for the region outside the well, is proportional to $(V_0 - E)^{1/2}$. It also agrees with the idea that the more serious the violation of the classical restriction, that the total energy E must be at least as large as the potential energy $V(x)$, the more reluctant the eigenfunctions are to penetrate the classically excluded regions.

It is instructive to consider the effect on the eigenfunctions of letting the walls of the square well become very high, i.e., letting $V_0 \rightarrow \infty$. Shown in Figure 6-27 is the first eigenfunction for a square well potential. As $V_0 \rightarrow \infty$, E_1 will increase, but it will do so very slowly compared to the increase in V_0 . This is true because E_1 is determined essentially by the requirement that approximately half an oscillation of the eigenfunction must fit into the length of the well. Therefore, the exponential parameter $k_{II} = \sqrt{2m(V_0 - E)}/\hbar$, which determines the behavior of the eigenfunction in the regions outside of the well, will become very large as V_0 becomes very large, and the eigenfunction will go to zero very rapidly outside the well. In the limit, $\psi_1(x)$ must be zero for all $x < -a/2$ and for all $x > +a/2$. For a square well with infinitely high walls, $\psi_1(x)$ has the form shown in Figure 6-28. It is apparent that this argument holds for all the eigenfunctions of such a potential. That is, for all values of n , in an infinite square well potential

$$\psi_n(x) = 0 \quad x \leq -a/2 \text{ or } x \geq +a/2 \quad (6-65)$$

This condition for infinite square well eigenfunctions can only be satisfied by violating at $x = \pm a/2$ the requirement of Section 5-6 that the derivative $d\psi_n(x)/dx$ of an eigenfunction be continuous everywhere; but if the student will review the argument which

**FIGURE 6-28**

The first eigenfunction of a square well with walls of infinite height.

was presented to justify the requirement, he will find that the derivative must be continuous only when the potential is finite.

6-8 The Infinite Square Well Potential

The infinite square well potential is written as

$$V(x) = \begin{cases} \infty & x < -a/2 \text{ or } x > +a/2 \\ 0 & -a/2 < x < +a/2 \end{cases} \quad (6-66)$$

and is illustrated in Figure 6-29. It has the feature that it will bind a particle with any finite total energy $E \geq 0$. In *classical mechanics*, any of these energies are possible, but in *quantum mechanics* only *certain* discrete eigenvalues E_n are allowed.

We shall see that it is very easy to find simple and concise expressions for the eigenvalues and eigenfunctions of this potential because the transcendental equation that arises in the solution of its time-independent Schroedinger equation happens to have simple solutions. For values of the quantum number n which are not too large, these eigenvalues and eigenfunctions can often be used to approximate the corresponding (same n) eigenvalues and eigenfunctions of a square well potential with large but finite V_0 . For instance, we mentioned before that it is a very good approximation to take the potential for a conduction electron in a block of metal to be a finite square well. In Example 6-2 we showed that for the typical metal Cu the eigenfunctions penetrate into the classically excluded regions exterior to the well by a distance of about 10^{-10} m. This distance is so small compared to the width of the square well, which is the width of the Cu block, that for many purposes it is an equally good approximation to use the corresponding eigenfunctions and eigenvalues for an infinite square well, and we shall do so later. We shall also use infinite square well potentials to discuss the quantum mechanical properties of a system of gas molecules, and other particles, which are strictly confined within a box of certain dimensions. A particle moving under the influence of an infinite square well potential is often called a *particle in a box*.

In the region within the well the *general solution* to the time-independent Schroedinger equation for the infinite square well potential can be written as the standing wave of (6-62), which we simplify, by dropping the primes, into the form

$$\psi(x) = A \sin kx + B \cos kx \quad \text{where } k = \frac{\sqrt{2mE}}{\hbar} \quad -a/2 \leq x \leq +a/2 \quad (6-67)$$

(Students who have skipped the preceding sections can see that this $\psi(x)$ represents a standing wave by noting that the associated wave function $\Psi(x,t) = \psi(x)e^{-iEt/\hbar}$ has fixed nodes. They can verify that the $\psi(x)$ is actually a solution to the applicable

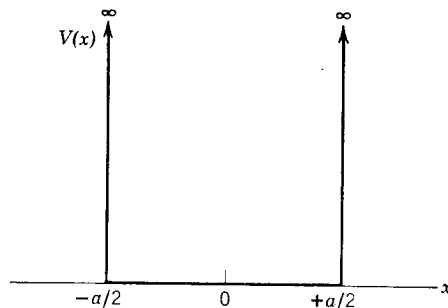


FIGURE 6-29

An infinite square well potential.

time-independent Schroedinger equation by substituting it into (6-2).) According to the condition of (6-65), $\psi(x)$ has the value zero in the regions outside the well. Of course, this must be true so that the probability density will be zero in these regions, since the particle is strictly confined within the well by its infinitely high potential walls. In particular, at the boundaries of the well

$$\psi(x) = 0 \quad x = \pm a/2 \quad (6-68)$$

That is, the standing wave has nodes at the walls of the box.

Now we develop relations which are satisfied by the arbitrary constants A and B , and by the parameter k . Applying the boundary conditions of (6-68) at $x = +a/2$, we obtain

$$A \sin \frac{ka}{2} + B \cos \frac{ka}{2} = 0 \quad (6-69)$$

At $x = -a/2$, (6-68) yields

$$A \sin \left(-\frac{ka}{2} \right) + B \cos \left(-\frac{ka}{2} \right) = 0$$

or

$$-A \sin \frac{ka}{2} + B \cos \frac{ka}{2} = 0 \quad (6-70)$$

Addition of the last two numbered equations gives

$$2B \cos \frac{ka}{2} = 0 \quad (6-71)$$

Subtraction gives

$$2A \sin \frac{ka}{2} = 0 \quad (6-72)$$

Both (6-71) and (6-72) must be satisfied. When this is done, $\psi(x)$ and $d\psi(x)/dx$ will be everywhere finite and single valued, and $\psi(x)$ will be everywhere continuous. As discussed at the end of the preceding section, $d\psi(x)/dx$ will be discontinuous at $x = \pm a/2$.

There is no value of the parameter k for which both $\cos(ka/2)$ and $\sin(ka/2)$ are simultaneously zero. And we certainly do not want to satisfy the two equations by setting both A and B equal to zero, for then $\psi(x) = 0$ everywhere and the eigenfunction would be of no interest because the associated particle would not be in the box! However, we can satisfy these equations *either* by choosing k so that $\cos(ka/2)$ is zero and also setting A equal to zero, *or* by choosing k so that $\sin(ka/2)$ is zero and also setting B equal to zero. That is, we take either

$$A = 0 \quad \text{and} \quad \cos \frac{ka}{2} = 0 \quad (6-73)$$

or

$$B = 0 \quad \text{and} \quad \sin \frac{ka}{2} = 0 \quad (6-74)$$

Thus there are two classes of solutions.

For the *first class*

$$\psi(x) = B \cos kx \quad \text{where} \quad \cos \frac{ka}{2} = 0 \quad (6-75)$$

For the *second class*

$$\psi(x) = A \sin kx \quad \text{where} \quad \sin \frac{ka}{2} = 0 \quad (6-76)$$

The conditions on the wave number k , expressed in (6-75) and (6-76), are in the form of transcendental equations since the unknown, k , occurs in the arguments of the sinusoids; but these transcendental equations happen to be so simple that their solutions can be written in concise form immediately. The allowed values of k for the first class, (6-75), are

$$\frac{ka}{2} = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \dots$$

since $\cos(\pi/2) = \cos(3\pi/2) = \cos(5\pi/2) = \dots = 0$. It is convenient to express this as

$$k_n = \frac{n\pi}{a} \quad n = 1, 3, 5, \dots \quad (6-77)$$

The allowed values of k for the second class, (6-76), are

$$\frac{ka}{2} = \pi, 2\pi, 3\pi, \dots$$

since $\sin \pi = \sin 2\pi = \sin 3\pi = \dots = 0$. This can also be expressed as

$$k_n = \frac{n\pi}{a} \quad n = 2, 4, 6, \dots \quad (6-78)$$

Knowing the allowed values of k , we can then obtain the solutions to the time-independent Schroedinger equation for the infinite square well from (6-75) and (6-76). We find

$$\psi_n(x) = B_n \cos k_n x \quad \text{where } k_n = \frac{n\pi}{a} \quad n = 1, 3, 5, \dots \quad (6-79)$$

and

$$\psi_n(x) = A_n \sin k_n x \quad \text{where } k_n = \frac{n\pi}{a} \quad n = 2, 4, 6, \dots \quad (6-80)$$

The solution corresponding to $n = 0$ is $\psi_0(x) = A \sin 0 = 0$; it is ruled out because it does not describe a particle in a box. The quantum number n has been used to label the different solutions of the transcendental equations, and the corresponding eigenfunctions. If it is necessary to apply the normalization condition, the constants A_n and B_n , which specify the amplitudes of the eigenfunctions, will thereby be determined (see Example 5-10); but it is not usually necessary to do this.

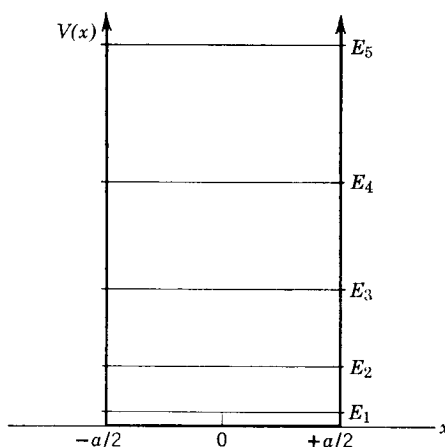
The quantum number n is also used to label the corresponding eigenvalues. Using the relation $k = \sqrt{2mE}/\hbar$ of (6-67), and the expression $k_n = n\pi/a$ in (6-79) and (6-80) for the allowed values of k , we find

$$E_n = \frac{\hbar^2 k_n^2}{2m} = \frac{\pi^2 \hbar^2 n^2}{2ma^2} \quad n = 1, 2, 3, 4, 5, \dots \quad (6-81)$$

Thus we conclude that only certain values of the total energy E are allowed. The total energy of the particle in the box is *quantized*.

The quantitative treatment of the finite square well, discussed in the preceding section and carried out in Appendix G, is essentially the same as what we have just gone through. But the penetration of the eigenfunction into the regions outside the well, which varies with the energy of the associated eigenvalue, leads to more complicated transcendental equations for k that must be solved graphically or numerically.

Figure 6-30 illustrates the infinite square well potential and its first few eigenvalues specified by (6-81). Of course, all the eigenvalues are discretely separated for an

**FIGURE 6-30**

The first few eigenvalues of an infinite square well potential.

infinite square well potential since the particle is bound for any finite eigenvalue. Note that the pattern formed by the first three eigenvalues of the infinite square well is quite similar to that formed by the three bound eigenvalues of the finite square well shown in Figure 6-25. In this regard, the infinite square well results provide an approximation to the finite square well results. However, in detail each potential energy function $V(x)$ has its own characteristic set of bound eigenvalues E_n .

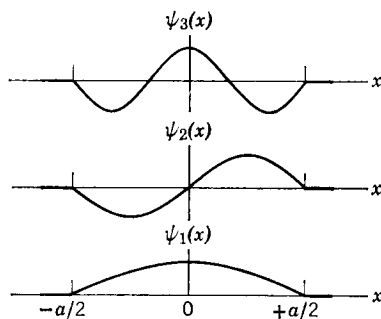
Of particular interest is the energy of the first eigenvalue. For the infinite square well it is

$$E_1 = \frac{\pi^2 \hbar^2}{2ma^2} \quad (6-82)$$

This is called the *zero-point energy*. It is the lowest possible total energy the particle can have if it is bound by the infinite square well potential to the region $-a/2 \leq x \leq +a/2$. *The particle cannot have zero total energy.* The phenomenon is basically a result of the uncertainty principle. To see this, consider the fact that if the particle is bound by the potential, then we know its x coordinate to within an uncertainty of about $\Delta x \simeq a$. Consequently, the uncertainty in its x momentum must be at least $\Delta p \simeq \hbar/2\Delta x \simeq \hbar/2a$. The uncertainty principle cannot allow the particle to be bound by the potential with zero total energy since that would mean the uncertainty in the momentum would be zero. For the particular case of eigenvalue E_1 , the magnitude of the momentum is $p_1 \simeq \sqrt{2mE_1} = \pi\hbar/a$. Since the particle is in a state of motion described by a standing wave eigenfunction, it can be moving in either direction and the actual value of the momentum is uncertain by an amount which is about $\Delta p \simeq 2p_1 \simeq 2\pi\hbar/a$. The uncertainty product $\Delta x \Delta p \simeq a(2\pi\hbar/a) \simeq 2\pi\hbar$ is roughly in agreement with the lower limit $\hbar/2$ set by the uncertainty principle. (Compare with the accurate calculation of Example 5-10.)

We conclude that there must be a zero-point energy because there must be a *zero-point motion*. This is in sharp contrast to the idea, of classical physics, that all motion ceases when a system has its minimum energy content at the temperature of absolute zero. The zero-point energy is responsible for several interesting quantum phenomena that are seen in the behavior of matter at very low temperatures. A striking example is the fact that helium will not solidify even at the lowest attainable temperature ($\sim 0.001^\circ\text{K}$).

The first few eigenfunctions of the infinite square well potential are shown in Figure 6-31. Note that the number of half wavelengths of each eigenfunction is equal to its

**FIGURE 6-31**

The first few eigenfunctions of an infinite square well potential.

quantum number n , and that therefore the number of nodes is $n + 1$. By comparing these eigenfunctions with the corresponding eigenfunctions of the finite square well shown in Figure 6-26, the student can see again how the results obtained for the simple potential can be used to approximate those of the more complicated potential (most accurately for eigenfunctions of lowest n value).

Students familiar with stringed musical instruments may notice that the eigenfunctions for a particle strictly confined between two points at the ends of the box look like the functions describing the possible shapes assumed by a vibrating string fixed at two points at the ends of the string. The reason is that both systems obey time-independent differential equations of analogous form, and they satisfy analogous conditions at the two points. Here is yet another example of the relation between quantum mechanics and classical wave motion. Musically inclined students may also notice that the frequencies, $\nu_n = E_n/h$, of the time-dependent factor in the wave functions for the confined particle satisfy the relation $\nu_n \propto n^2$ (since $E_n = \pi^2 \hbar^2 n^2 / 2ma^2$), whereas the frequencies of the vibrating string satisfy the “harmonic progression” $\nu_n \propto n$. This difference arises because the two systems obey time-dependent differential equations which are not at all analogous.

Example 6-5. Derive the infinite square well energy quantization law, (6-81), directly from the de Broglie relation $p = h/\lambda$, by fitting an integral number of half de Broglie wavelengths $\lambda/2$ into the width a of the well.

It is clear from Figure 6-31 that the infinite square well eigenfunctions satisfy the following relation between the de Broglie wavelengths and the length of the well

$$n \frac{\lambda}{2} = a \quad n = 1, 2, 3, \dots$$

That is, an integral number of half-wavelengths fits into the well. This means

$$\lambda = \frac{2a}{n} \quad n = 1, 2, 3, \dots$$

So according to de Broglie, the corresponding values of the momentum of the particle are

$$p = \frac{h}{\lambda} = \frac{hn}{2a} \quad n = 1, 2, 3, \dots$$

As the potential energy of the particle is zero within the well, its total energy equals its kinetic energy. Thus

$$E = \frac{p^2}{2m} = \frac{h^2 n^2}{2m4a^2} = \frac{\pi^2 \hbar^2 n^2}{2ma^2} \quad n = 1, 2, 3, \dots$$

in agreement with (6-81). This trivial calculation can be used only for the simplest case of a bound particle—the case of an infinite square well potential. It cannot be applied to find the

eigenvalues or eigenfunctions of a more complicated potential such as a finite square well. (See also the discussion, in connection with (4-25), of the application of the Wilson-Sommerfeld quantization rule to the infinite square well.) ◀

Example 6-6. Before the discovery of the neutron, it was thought that a nucleus of atomic number Z and atomic weight A was composed of A protons and $(A - Z)$ electrons, but there was a serious problem concerning the magnitude of the zero-point energy for a particle as light as an electron confined to a region as small as a nucleus. Estimate the zero-point energy E .

Setting the electron mass m equal to 10^{-30} kg and the width of the well equal to 10^{-14} m (a typical nuclear dimension), from (6-82) we obtain

$$\begin{aligned} E &= \frac{\pi^2 \hbar^2}{2ma^2} \simeq \frac{10 \times 10^{-68} \text{ joule}^2\text{-sec}^2}{2 \times 10^{-30} \text{ kg} \times 10^{-28} \text{ m}^2} \simeq \frac{10^{-9}}{2} \text{ joule} \\ &\simeq \frac{10^{-9} \text{ joule}}{2} \times \frac{1 \text{ eV}}{1.6 \times 10^{-19} \text{ joule}} \simeq 3 \times 10^9 \text{ eV} \\ &= 3000 \text{ MeV} \end{aligned}$$

For estimating the zero-point energy, we are certainly justified in treating the electron as if it were confined to an infinite square well. We are also justified in ignoring the three-dimensional character of the actual system. But we would not be justified in quoting the value of E just obtained because it is extremely large compared to the electron rest mass energy $m_0 c^2 \simeq 0.5 \text{ MeV}$. A relativistically valid analogue of (6-82) must be used in this particular problem.

The required formula can be obtained from the technique used in Example 6-5. Both of the equations $\lambda = 2a/n$ and $p = \hbar/\lambda$ retain their validity in the extreme relativistic range. So, if we replace $E = p^2/2m$ by $E = cp$ (the energy-momentum relation $E^2 = c^2 p^2 + m_0^2 c^4$ in the limit $E \gg m_0 c^2$), we immediately obtain for $n = 1$

$$\begin{aligned} E = cp &= \frac{ch}{\lambda} = \frac{chn}{2a} = \frac{\pi \hbar n}{a} \simeq \frac{3 \times 3 \times 10^8 \text{ m/sec} \times 10^{-34} \text{ joule-sec}}{10^{-14} \text{ m}} \\ &\times \frac{1 \text{ eV}}{1.6 \times 10^{-19} \text{ joule}} \simeq 6 \times 10^7 \text{ eV} = 60 \text{ MeV} \end{aligned}$$

An electron *could* be bound in a nucleus with this zero-point energy, if the magnitude of the depth of the binding potential were greater than the magnitude of the zero-point energy. There is a binding potential acting on the electron due to the Coulomb attraction of the positive charge of the nucleus, but the magnitude of the potential is not great enough. We may estimate this magnitude by setting $r = 10^{-14}$ m, and $Q_1 = Ae$, $Q_2 = -e$, where e is the magnitude of the electron charge, in the Coulomb potential formula. We obtain, for a typical value of $A = 100$

$$\begin{aligned} \frac{Q_1 Q_2}{4\pi\epsilon_0 r} &= -\frac{Ae^2}{4\pi\epsilon_0 r} \simeq -\frac{10^2 \times (1.6 \times 10^{-19} \text{ coul})^2}{10^{-10} \text{ coul}^2/\text{nt-m}^2 \times 10^{-14} \text{ m}} \times \frac{1 \text{ eV}}{1.6 \times 10^{-19} \text{ joule}} \\ &\simeq -1.6 \times 10^7 \text{ eV} = -16 \text{ MeV} \end{aligned}$$

This is four times smaller than the required binding energy. So an electron could not be bound in a nucleus because of the zero-point energy required by the uncertainty principle.

In 1932 Chadwick, motivated by a suggestion of Rutherford, discovered the neutron. We now know that a nucleus is composed of Z protons and $(A - Z)$ neutrons. Because neutrons are heavy particles, like protons, their zero-point energy in a nucleus is relatively low so they can be bound without difficulty. Indeed, we shall see in Chapter 15 that some of the most important properties of nuclei can be explained in terms of the quantum states of neutrons, and protons, moving in a (finite) square well potential. ◀

Figure 6-31 makes quite apparent the essential difference between the two classes of standing wave eigenfunctions specified by (6-79) and (6-80). The eigenfunctions

of the first class, $\psi_1(x)$, $\psi_3(x)$, $\psi_5(x)$, . . . , are *even functions* of x ; that is

$$\psi(-x) = +\psi(x) \quad (6-83)$$

In quantum mechanics, these functions are said to be an *even parity*. The eigenfunctions of the second class, $\psi_2(x)$, $\psi_4(x)$, $\psi_6(x)$, . . . , are *odd functions* of x ; that is

$$\psi(-x) = -\psi(x) \quad (6-84)$$

and are said to be of *odd parity*.

The eigenfunctions have a *definite parity*, either even or odd, because we have chosen the origin of the x axis so that the *symmetrical* square well potential $V(x)$ is an *even* function of x . Note that if we redefine the origin of the x axis in Figure 6-31 to be at, say, the point $x = -a/2$, the eigenfunctions will no longer have a definite parity.

These results are obtained for the square well potential, and for any other symmetrical potential, since measurable quantities describing the motion of a particle in *bound* states of such potentials must also be symmetrical about the point of symmetry of the potential. If the origin of the x axis is chosen to be at that symmetry point, then the function describing the measurable quantity must be an even function. As an example, this is true for the probability density function $P(x,t)$, for both even and odd parity eigenfunctions, since

$$P(-x,t) = \psi^*(-x)\psi(-x) = [\pm\psi^*(x)][\pm\psi(x)] = \psi^*(x)\psi(x) = P(x,t) \quad (6-85)$$

This is not true for the wave function itself in the case of an odd parity eigenfunction; such a wave function is an odd function of x , but this is not a contradiction because the wave function itself is not measurable. Eigenfunctions for *unbound* states of potentials that are even functions of x do not necessarily have definite parities since they do not necessarily describe symmetrical motions of the particle.

In one dimension, the fact that standing wave eigenfunctions have definite parities, if $V(-x) = V(x)$, is of importance largely because it simplifies certain calculations. In three dimensions, the property has a deeper significance that will be seen first in Chapter 8 in connection with the emission of radiation by an atom making a transition from an excited state to its ground state.

The probability density functions, corresponding to the first few eigenfunctions of the infinite square well, are plotted in Figure 6-32. Also illustrated in the figure is the probability density that would be predicted by classical mechanics for a bound particle bouncing back and forth between $-a/2$ and $+a/2$. Since the classical particle would spend an equal amount of time in any element of the x axis in that region, it would be equally likely found in any such element. The quantum mechanical probability density oscillates more and more as n increases. In the limit that n approaches infinity, that is for eigenvalues of very high energy, the oscillations are so compressed that no experiment could possibly have the resolution to observe anything other than the average behavior of the probability density predicted by quantum mechanics.

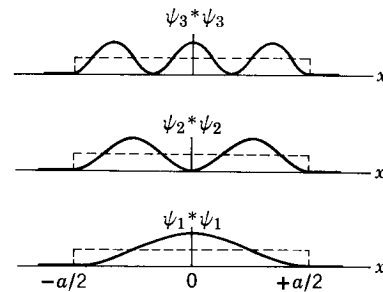


FIGURE 6-32

The first few probability density functions for an infinite square well potential. The dashed curves are the predictions of classical mechanics.

Furthermore, the *fractional* separation of the eigenvalues approaches zero as n approaches infinity, so in that limit their discreteness cannot be resolved. Thus we see that the quantum mechanical predictions approach the predictions of classical mechanics in the large quantum number, or high-energy, limit. This is what would be expected from the correspondence principle of the old quantum theory.

6-9 The Simple Harmonic Oscillator Potential

We have discussed several potentials which are discontinuous functions of position with constant values in adjacent regions. Now we turn to the more realistic cases of potentials which are continuous functions of position. It turns out that there are only a limited number of such potentials for which it is possible to obtain solutions to the Schrodinger equation by analytical techniques. But, fortunately, these potentials include some of the most important cases, such as the Coulomb potential, $V(r) \propto r^{-1}$, discussed in the following chapter, and the simple harmonic oscillator potential, $V(x) \propto x^2$, discussed in this section. (In this connection, we should remind the student that solutions to the Schrodinger equations for potentials of *any* form can always be obtained by the numerical techniques developed in Appendix F.)

The simple harmonic oscillator is of tremendous importance in physics, and all fields based on physics, because it is the prototype for any system involving oscillations. For instance, it is used in the study of: the vibration of atoms in diatomic molecules, the acoustic and thermal properties of solids which arise from atomic vibrations, magnetic properties of solids that involve vibrations in the orientation of nuclei, and the electrodynamics of quantum systems in which electromagnetic waves are vibrating. Generally speaking, the simple harmonic oscillator can be used to describe almost any system in which an entity is executing *small vibrations* about a point of *stable equilibrium*.

At a position of stable equilibrium, the potential function $V(x)$ must have a minimum. Since any realistic potential function is continuous, the function in the region near its minimum can almost always be well approximated by a parabola, as illustrated in Figure 6-33. But for *small* vibrations the only thing that counts is what $V(x)$ does near its minimum. If we choose the origins of the x axis and the energy axis to be at the minimum, we can write the equation for this parabolic potential function as

$$V(x) = \frac{C}{2} x^2 \quad (6-86)$$

where C is a constant. Such a potential is illustrated in Figure 6-34. A particle moving under its influence experiences a linear (or Hooke's law) restoring force $F(x) = -dV(x)/dx = -Cx$, with C being the force constant.

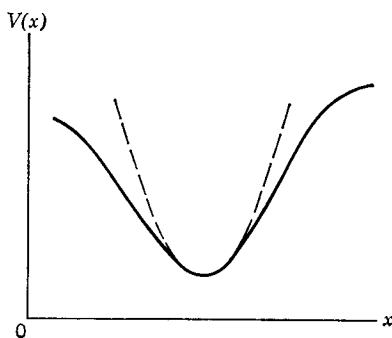
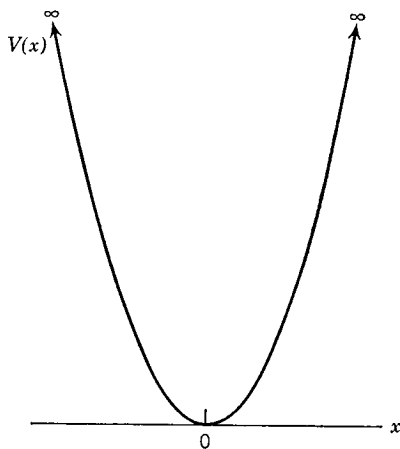


FIGURE 6-33

Illustrating the fact that any continuous potential with a minimum (solid curve) can be approximated near the minimum very well by a parabolic potential (dashed curve).

**FIGURE 6-34**

The simple harmonic oscillator potential.

Classical mechanics predicts that a particle under the influence of the linear restoring force exerted by the potential of (6-86), which is displaced by an amount x_0 from the equilibrium position and then released, will oscillate in simple harmonic motion about the equilibrium position with frequency

$$\nu = \frac{1}{2\pi} \sqrt{\frac{C}{m}} \quad (6-87)$$

where m is its mass. According to that theory, the total energy E of the particle is proportional to x_0^2 , and can have *any* value since x_0 is arbitrary.

Quantum mechanics predicts that the total energy E can assume *only* a discrete set of values because the particle is bound by the potential to a region of finite extent. Even in the old quantum theory this was known. The student will recall that Planck's postulate predicts the energy of a particle executing simple harmonic oscillations can assume only one of the values

$$E_n = nh\nu \quad n = 0, 1, 2, 3, \dots \quad (6-88)$$

What are the allowed energy values predicted by Schroedinger quantum mechanics for this very important potential? To find out, the time-independent Schroedinger equation for the simple harmonic oscillator potential must be solved.

The mathematics used in the analytical solution to the equation is not difficult to follow, and it is quite interesting; but since the solution is very lengthy it has been placed in Appendix H. Other than verifying by substitution a typical eigenfunction and eigenvalue obtained from the solution, here we shall concentrate on describing the results of the solution and discussing their physical significance.

It is found that the eigenvalues for the simple harmonic oscillator potential are given by the formula

$$E_n = (n + 1/2)h\nu \quad n = 0, 1, 2, 3, \dots \quad (6-89)$$

where ν is the classical oscillation frequency of the particle in the potential. All the eigenvalues are discrete since the particle is bound for any of them. The potential, and the eigenvalues, are shown in Figure 6-35.

If we compare the Schroedinger results with the Planck postulate, we see that in quantum mechanics all the eigenvalues are shifted up by an amount $h\nu/2$. As a consequence, the minimum possible total energy for a particle bound to the potential is $E_0 = h\nu/2$. This is the zero-point energy for the potential, the existence of which is required by the uncertainty principle. Therefore, Planck's postulated energy

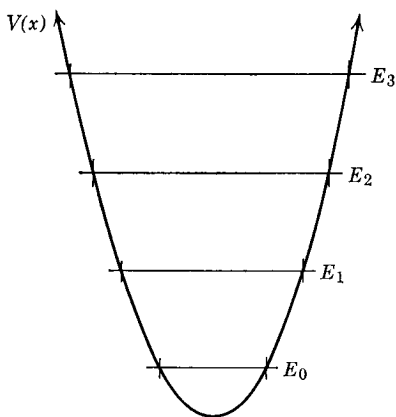


FIGURE 6-35

The first few eigenvalues of the simple harmonic oscillator potential. Note that the classically allowed regions (between the intersections of $V(x)$ and E_n) expand with increasing values of E_n .

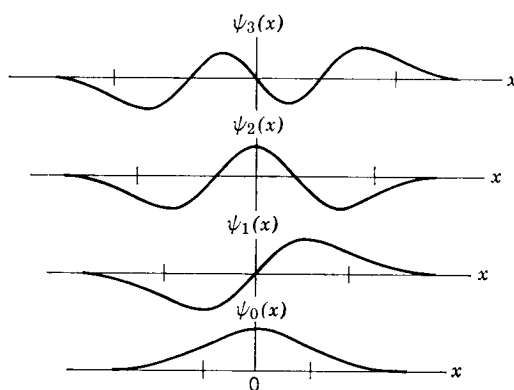
quantization of the simple harmonic oscillator was actually in error by the additive constant $h\nu/2$. This constant cancels out in most applications of Planck's postulate because they involve only differences between two energy values. As an example, consider the electromagnetic radiation emitted by the vibrating charge distribution of a diatomic molecule whose interatomic spacing is executing simple harmonic oscillations. Since the frequencies of the emitted photons depend only on the differences in the allowed energies of the molecule, the additive constant has no effect on the *frequencies* of the photons.

But there are observable quantities that show Planck's postulate is in error because it does not contain the zero-point energy. The most important example is also connected with the emission of radiation by a vibrating molecule, or atom. When we study this subject in a subsequent chapter, we shall see that the *rate of emission* of the photons would not agree with experiment unless simple harmonic oscillators have zero-point energies. In fact, we shall find the only reason why the molecule emits *any* radiation is that its vibrations have been stimulated by a surrounding electromagnetic field whose field strengths are executing simple harmonic oscillations because of the zero-point energy of the field.

In addition to providing completely correct eigenvalues, quantum mechanics also provides the eigenfunctions for the simple harmonic oscillator. The eigenfunctions ψ_n , corresponding to the first few eigenvalues E_n , are listed in Table 6-1 and plotted in Figure 6-36. The eigenfunctions are expressed in terms of the dimensionless variable $u = [(Cm)^{1/4}/\hbar^{1/2}]x$, which differs from x only by a proportionality constant that depends on the properties of the oscillator. For all values of x , the eigenfunction is

TABLE 6-1. Some Eigenfunctions $\psi(u)$ for the Simple Harmonic Oscillator Potential, where u is Related to the Coordinate x by the Equation $u = [(Cm)^{1/4}/\hbar^{1/2}]x$

| Quantum Number | Eigenfunctions |
|----------------|--|
| 0 | $\psi_0 = A_0 e^{-u^2/2}$ |
| 1 | $\psi_1 = A_1 u e^{-u^2/2}$ |
| 2 | $\psi_2 = A_2 (1 - 2u^2) e^{-u^2/2}$ |
| 3 | $\psi_3 = A_3 (3u - 2u^3) e^{-u^2/2}$ |
| 4 | $\psi_4 = A_4 (3 - 12u^2 + 4u^4) e^{-u^2/2}$ |
| 5 | $\psi_5 = A_5 (15u - 20u^3 + 4u^5) e^{-u^2/2}$ |

**FIGURE 6-36**

The first few eigenfunctions of the simple harmonic oscillator potential. The vertical ticks on the x axes indicate the limits of classical motion shown in Figure 6-35.

given by the product of an exponential, whose exponent is proportional to $-x^2$, times a simple polynomial of order x^n . The polynomial is responsible for the oscillatory behavior of ψ_n in the classically allowed region where $E_n < V(x)$. The number of oscillations increases with increasing n because there are n values of x for which a polynomial of the order x^n has the value zero. These values of x are the locations of the nodes of ψ_n . The classically allowed regions lie within the vertical marks shown in Figure 6-36. These regions become wider with increasing n because of the shape of the simple harmonic oscillator potential $V(x)$, as can be seen by inspecting Figure 6-35 which also indicates the classically allowed regions for each E_n . Outside these regions, the eigenfunctions decrease very rapidly because their behavior is dominated by the decreasing exponential. Since the relation $V(-x) = V(x)$ is satisfied by the potential, we expect that its eigenfunctions should have definite parities. Inspection of Table 6-1 shows this is true, and that the parity is even for even n and odd for odd n . Thus the eigenfunction for the lowest allowed energy is of even parity, as in the case of a square well potential. The multiplicative constants A_n determine the amplitudes of the eigenfunctions. If necessary, the normalization procedure can be used to fix their values, as in Example 5-7; but this is usually not necessary.

The simple harmonic oscillator eigenfunctions contain a wealth of information about the behavior of the system. Some of this information was extracted in Chapter 5. For instance, Figures 5-3 and 5-18 gave accurate representations of the probability density functions for the $n = 1$ and $n = 13$ quantum states of the oscillator. In Chapter 8 we shall show how the eigenfunctions can be used to calculate the rate of emission of radiation by a charged simple harmonic oscillator, and derive the $n_i - n_f = \pm 1$ selection rule that had to be introduced in the old quantum theory by arguments based on the rather unreliable correspondence principle.

Example 6-7. Because the simple harmonic oscillator eigenfunctions for small n have fairly simple mathematical forms, it is not too difficult to verify by direct substitution that they satisfy the time-independent Schroedinger equation, for the potential of (6-86), and for the eigenvalues of (6-89). Make such a verification for $n = 1$. (For $n = 0$ the wave function was verified by direct substitution in the Schroedinger equation in Example 5-3.)

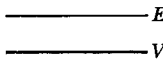
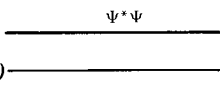
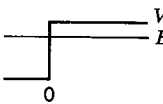
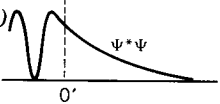
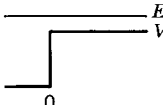
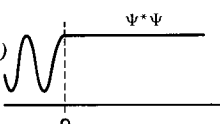
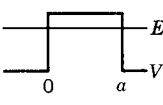
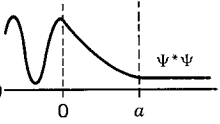

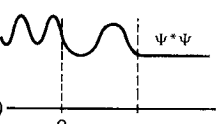
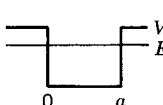
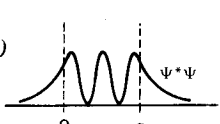
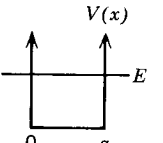
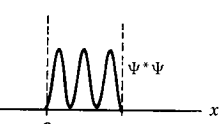
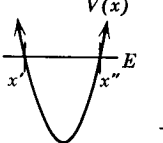
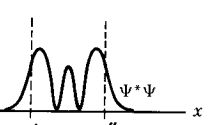
The time-independent Schroedinger equation is

$$-\frac{\hbar^2}{2m} \frac{d^2\psi}{dx^2} + \frac{C}{2} x^2 \psi = E\psi$$

To verify that the eigenvalue

$$E_1 = \frac{3}{2} h\nu = \frac{3}{2} \frac{h}{2\pi} \left(\frac{C}{m} \right)^{1/2} = \frac{3}{2} \hbar \left(\frac{C}{m} \right)^{1/2}$$

TABLE 6-2. A Summary of the Systems Studied in Chapter 6

| Name of System | Physical Example | Potential and Total Energies | Probability Density | Significant Feature |
|--------------------------------------|--|---|---|---|
| Zero potential | Proton in beam from cyclotron |  |  | Results used for other systems |
| Step potential (energy below top) | Conduction electron near surface of metal |  |  | Penetration of excluded region |
| Step potential (energy above top) | Neutron trying to escape nucleus |  |  | Partial reflection at potential discontinuity |
| Barrier potential (energy below top) | α particle trying to escape Coloumb barrier |  |  | Tunneling |
| Barrier potential (energy above top) | Electron scattering from negatively ionized atom |  |  | No reflection at certain energies |
| Finite square well potential | Neutron bound in nucleus |  |  | Energy quantization |
| Infinite square well potential | Molecule strictly confined to box |  |  | Approximation to finite square well |
| Simple harmonic oscillator potential | Atom of vibrating diatomic molecule |  |  | Zero-point energy |

and the eigenfunction

$$\psi_1 = A_1 u e^{-u^2/2} \quad \text{where } u = \frac{(Cm)^{1/4}}{\hbar^{1/2}} x$$

satisfy the equation, we evaluate the derivatives

$$\begin{aligned} \frac{d\psi_1}{dx} &= \frac{du}{dx} \frac{d\psi_1}{du} = \frac{(Cm)^{1/4}}{\hbar^{1/2}} [A_1 e^{-u^2/2} + A_1 u(-u)e^{-u^2/2}] \\ &= \frac{(Cm)^{1/4}}{\hbar^{1/2}} A_1 e^{-u^2/2} [1 - u^2] \end{aligned}$$

and

$$\begin{aligned} \frac{d^2\psi_1}{dx^2} &= \frac{du}{dx} \frac{d}{du} \frac{d\psi_1}{dx} = \frac{(Cm)^{1/4}}{\hbar^{1/2}} \left\{ \frac{d}{du} \frac{(Cm)^{1/4}}{\hbar^{1/2}} A_1 e^{-u^2/2} [1 - u^2] \right\} \\ &= \frac{(Cm)^{1/2}}{\hbar} A_1 \{-u e^{-u^2/2} [1 - u^2] + e^{-u^2/2} [-2u]\} \\ &= \frac{(Cm)^{1/2}}{\hbar} A_1 u e^{-u^2/2} \{u^2 - 3\} \\ &= \frac{(Cm)^{1/2}}{\hbar} \{u^2 - 3\} \psi_1 = \frac{(Cm)^{1/2}}{\hbar} \left\{ \frac{(Cm)^{1/2}}{\hbar} x^2 - 3 \right\} \psi_1 \end{aligned}$$

Substitution of $d^2\psi_1/dx^2$ and E_1 into the equation they are supposed to satisfy yields

$$-\frac{\hbar^2}{2m} \frac{(Cm)^{1/2}}{\hbar} \left\{ \frac{(Cm)^{1/2}}{\hbar} x^2 - 3 \right\} \psi_1 + \frac{C}{2} x^2 \psi_1 = \frac{3\hbar}{2} \left(\frac{C}{m} \right)^{1/2} \psi_1$$

Since inspection shows this is satisfied, the verification is completed. ◀

6-10 Summary

In Table 6-2 we summarize some of the properties of the systems studied in this chapter. The table gives an abbreviated name for each idealized system, and an example of a physical system whose potential and total energies are approximated by the idealization. It also gives sketches of the forms of the potential and total energies, and corresponding probability density functions, for each system. If the particle is not bound, it is incident from the left. We have chosen one significant feature of each system to list in the table, but there are many other significant features that we have discussed, which are not listed. In fact, in this chapter we have obtained most of the important predictions of quantum mechanics for systems involving one particle moving in a one-dimensional potential. In the following chapters we shall obtain predictions from the theory for systems involving three dimensions and several particles.

QUESTIONS

1. Can there be solutions with $E < 0$ to the time-independent Schroedinger equation for the zero potential?
2. Why is it never possible in classical mechanics to have $E < V(x)$? Why is it possible in quantum mechanics, providing there is some region in which $E > V(x)$?

3. Explain why the general solution to a one-dimensional time-independent Schrodinger equation contains two different functions, while the general solution to the corresponding Schrodinger equation contains many different functions.
4. Consider a particle in a long beam of very accurately known momentum. Does a wave function in the form of a group provide a more or a less realistic description of the particle than a single complex exponential wavefunction like (6-9)?
5. Under what circumstances is a discontinuous potential function a reasonable approximation to an actual system?
6. If a potential function has a discontinuity at a certain point, do its eigenfunctions have discontinuities at that point? If not, why not?
7. By combining oppositely directed traveling waves of equal amplitudes, we obtain a standing wave. What kind of a wave do we get if the amplitudes are not equal?
8. Just what is a probability flux, and why is it useful?
9. How can it be that a probability flux is split at a potential discontinuity, although the associated particle is not split?
10. Is there an analogy between the splitting of a probability flux that characterizes the behavior of an unbound particle in a one-dimensional system, and the alternative paths that can be followed by an unbound particle moving in two dimensions through a diffraction apparatus? Why?
11. Exactly what is meant by the statement that the reflection coefficient is one for a particle incident on a potential step with total energy less than the step height? What is meant by the statement that the reflection coefficient is less than one if the total energy is greater than the step height? Can the reflection coefficient ever be greater than one?
12. Since a real exponential is a nonoscillatory function, why is a complex exponential an oscillatory function?
13. What do you think causes the rapid oscillations in the group wave function of Figure 6-8 as it reflects from the potential step?
14. What is the fallacy in the following statement? "Since a particle cannot be detected while tunneling through a barrier, it is senseless to say that the process actually happens."
15. A particle is incident on a potential barrier, with total energy less than the barrier height, and it is reflected. Does the reflection involve only the potential discontinuity facing its direction of incidence? If the other discontinuity were removed, so that the barrier were changed into a step, is the reflection coefficient changed?
16. In the sun, two nuclei of low mass in violent thermal motion can collide by penetrating the Coulomb barrier which separates them. The mass of the single nucleus formed is less than the sum of the masses of the two nuclei, so energy is liberated. This *fusion* process is responsible for the heat output of the sun. What would be the consequences to life on earth if it could not happen because barriers were impenetrable?
17. Are there any *measurable* consequences of the penetration of a classically excluded region which is of infinite length? Consider a bound particle in a finite square well potential.
18. Show from a qualitative argument that a one-dimensional finite square well potential always has one bound eigenvalue, no matter how shallow the binding region. What would the eigenfunction look like if the binding region were very shallow?
19. Why do finite square wells have only a finite number of bound eigenvalues? What are the characteristics of the unbound eigenvalues?
20. What would a standing wave eigenfunction for an unbound eigenvalue of a finite square well look like?